A JORDAN SPACE CURVE WHICH BOUNDS NO FINITE SIMPLY CONNECTED AREA

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The author, in collaboration with Ph. Franklin, has recently given a simple example of a Jordan space curve, in the form of a skew polygon of a denumerable infinity of sides, every surface bounded by which has infinite area.[1] Since then, I have found an even simpler and more striking example, which it is the purpose of the present note to communicate.

Description.—In a Cartesian xy-plane, take the segment of the x-axis between (3,0) and (4,0) as diameter, and construct the circle C_0. If this be subjected to a homothetic transformation from the origin O as pole and of ratio

$$\rho = \frac{1}{\sqrt{2}} < \frac{3}{4},$$

a circle C_1 is obtained whose disc is entirely external to that of C_0. By repeating the operation indefinitely, that is, effecting homothetic transformations from O of ratios

$$1, \rho, \rho^2, \ldots, \rho^n, \ldots$$

we obtain an infinite sequence of circles

$$C_0, C_1, C_2, \ldots, C_n, \ldots,$$

mutually external, and shrinking to the point O as limit.

The areas of these circles form a geometric progression of ratio $\rho^2 = \frac{1}{2}$, so that, since $C_0 = \frac{\pi}{4}$,

$$C_n = \left(\frac{1}{2}\right)^n \frac{\pi}{4},$$

or

$$2^n C_n = \frac{\pi}{4}. \tag{1}$$

Next, let the entire configuration of circles be revolved about the y-axis. Then we obtain a nest of similar tori

$$T_0, T_1, T_2, \ldots, T_n, \ldots,$$

mutually external, and shrinking to the point O as limit.

Now take a thread (of infinite length) and wind it successively, and always in the same sense, once about T_0, twice about T_1, four times about $T_2, \ldots, 2^n$ times about $T_n, \ldots$. The thread is supposed to be pulled taut between successive tori—for instance, in the form of a common external tangent to their circular sections by the same axial plane. The winding may be quite arbitrary, subject only to the condition that the thread shall not intersect itself. Since the thread remains, from a certain point on, within an arbitrarily small sphere about the point O as center, it is evident that this construction gives a Jordan arc going from (4,0,0) to (0,0,0)—being a one-one continuous image of a line segment, *end-points included.* If to this arc we adjoin the lower semi-circle in the xy-plane (supposed vertical) on the segment (0,0) to (4,0) as diameter, a Jordan space curve J is formed.

J is my example.

Proof.—For the proof that every simply connected surface bounded by J has infinite area, we need nothing more than the following lemma.

LEMMA. *If a closed contour links N times with a torus, the area of whose generating circle is C, then on every simply connected surface bounded by the contour, the torus intercepts an area at least equal to NC.*

The reader will probably be willing to accept this lemma as evident. A proof may be had by means of the formula

$$S = \int\int \sec \gamma d\sigma_m$$

for the area of any surface, where $d\sigma_m$ is the circular projection of an element $d\sigma$ of the surface on any plane through a certain fixed axis, and γ is the angle between the tangent plane to the surface and the axial plane through the point of contact. By circular projection, we mean using as projecting curves the circles produced by rotating space about the given axis.

If the axis of the torus recedes to infinity, the torus becomes a cylinder, circular projection becomes ordinary orthogonal projection and the above formula is the classic one for the area of a surface in rectangular coördinates.

To establish the lemma requires, besides the trivial remark $\sec \gamma \geqq 1$, only the obvious topological fact that every element $d\sigma_m$ of C corresponds to at least N elements $d\sigma$.

The condition of simple connectivity on the surface S is absolutely essential. If we form a Möbius strip from a rectangle in the usual way, making the construction so that the strip links with a given torus, then the boundary of the strip links twice with the torus, whereas the area of the strip intercepted by the torus is zero.

Now, let S denote any simply-connected surface bounded by J. Since the linking number of J with the torus T_n is $N = 2^n$, it follows by the lemma that the area of the portion of S intercepted by T is at least

$$2^n C_n = \frac{\pi}{4},$$

according to (1). But the number of tori is infinite, and they are mutually external; consequently

$$Area\ of\ S \geqq \frac{\pi}{4} + \frac{\pi}{4} + \ldots + \frac{\pi}{4} + \ldots,$$

or

$$Area\ of\ S = +\infty.$$

In my solution of the problem of Plateau,[2] I have shown that every Jordan curve is the boundary of a simply connected minimal surface M:

$$x_i = RF_i(w), \qquad \sum_{i=1}^{n} F_i'^2(w) = 0.$$

This applies to the present example J, where the area of M like that of every other simply connected surface bounded by J, is infinite. A good sense in which M still has the least area property, in spite of the area functional becoming identically infinite, was given by me in a previous paper.[3]

[1] "A Step-Polygon of a Denumerable Infinity of Sides Which Bounds No Finite Area," these PROCEEDINGS, **19**, 188–191 (1933).

[2] "Solution of the Problem of Plateau," *Trans. Amer. Math. Soc.*, **33**, No. 1, 263–321 (1931).

[3] "The Least Area Property of the Minimal Surface Determined by an Arbitrary Jordan Contour," these PROCEEDINGS, **17**, 211–216 (1931).

ON LACUNARY POWER SERIES

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1. We consider a power series

$$\sum_{k=1}^{\infty} a_k z^{n_k}, \tag{1}$$

where $n_{k+1}/n_k \geq q > 1$ $(k = 1, 2, \ldots)$, the circle of convergence is supposed to be $|z| = 1$, and

$$\sum_{k=1}^{\infty} |a_k| = \infty. \tag{2}$$

I have proved the following theorem, which was suggested to me by Professor Zygmund.